Solutions
Math 220
HW # 3
October 8, 2018

Exercise 1. *Use the logical equivalences* $P \Rightarrow Q \equiv \neg P \vee Q$ *and* $P \Leftrightarrow Q \equiv (\neg P \vee Q) \wedge (\neg Q \vee P)$ *to rewrite*

$$(P \Rightarrow Q) \Leftrightarrow (Q \Rightarrow R).$$

without $\Rightarrow$ *or* $\Leftrightarrow$.

Solution.

$$\begin{aligned} (P \Rightarrow Q) \Leftrightarrow (Q \Rightarrow R) &\equiv (\neg P \vee Q) \Leftrightarrow (\neg Q \vee R) \\ &\equiv (\neg(\neg P \vee Q) \vee (\neg Q \vee R)) \wedge ((\neg P \vee Q) \vee \neg(\neg Q \vee R)) \end{aligned}$$

□

Exercise 2. *Write the* **negation** *(without just putting the phrase "It is not the case that..." in front of the given phrase) and the* **contrapositive** *of each of the following statements.*

(*a*) *If today is New Year's Eve, then tomorrow is January.*
(*b*) *If the decimal expansion of* r *is terminating, then* r *is rational.*
(*c*) *If* n *is divisible by* 6, *then* n *is divisible by* 2 *and* n *is divisible by* 3.

Solution.

(*a*) (*i*) The negation is: Today is New Year's Eve and tomorrow is not January.
(*ii*) The contrapositive is: If tomorrow is not January, then today is not New Year's Eve.

(*b*) (*i*) The negation is: The decimal expansion of r is terminating and r is irrational.
(*ii*) The contrapositive is: If r is irrational, then the decimal expansion of r is not terminating.

(*c*) (*i*) The negation is: n is divisible by 6 and n is not divisible by 2 or 3.
(*ii*) The contrapositive is: If n is not divisible by 2 or 3, then n is not divisible by 6.

□

Exercise 3. *Let* $P(x) : x > 3$ *and* $Q(x) : 4x - 1 > 12$ *be open sentences with domain* $D = \{0, 2, 3, 4, 6\}$. *Determine the truth value of* $P(x) \Rightarrow Q(x)$ *for each* $x \in D$. *Give explanations for your answers.*

Solution. The implication $P(x) \Rightarrow Q(x)$ is true when $P(x)$ and $Q(x)$ are both true, or $P(x)$ is false. $P(x)$ is true for $x = 4, 6$ and false for $x = 0, 2, 3$. $Q(x)$ is true for $x = 4, 6$. Therefore $P(x) \Rightarrow Q(x)$ is true for all $x \in D$. □

Exercise 4. *Let $P(x) : x \geq 3$ and $Q(x) : 4x - 1 > 12$ be open sentences with domain $D = \{0, 2, 3, 4, 6\}$. Determine all values of $x \in D$ for which $P(x) \Leftrightarrow Q(x)$ is true. Give an explanation for your answer.*

Solution. The biconditional $P(x) \Leftrightarrow Q(x)$ is true when $P(x)$ and $Q(x)$ are both true, or both false. $P(x)$ is true for $x = 3, 4, 6$ and false for $x = 0, 2$. $Q(x)$ is true for $x = 4, 6$ and false for $x = 0, 2, 3$. Therefore $P(x) \Leftrightarrow Q(x)$ is true for $x \in D$, $x \neq 3$, and false for $x = 3$. □

Exercise 5. *Let $P(x) : x^2 \geq 1$ and $Q(x) : x \geq 1$ be open sentences with domain $D = \mathbb{R}$. Determine all $x \in D$ for which $P(x) \Rightarrow Q(x)$ is a true statement. Give an explanation for your answer.*

Solution. The implication $P(x) \Rightarrow Q(x)$ is true when $P(x)$ and $Q(x)$ are both true, or $P(x)$ is false. $P(x)$ is true on the intervals $(-\infty, -1]$ and $[1, \infty)$ and is false on the interval $(-1, 1)$. $Q(x)$ is true on the interval $[1, \infty)$. Therefore $P(x) \Rightarrow Q(x)$ is true on the interval $(-1, \infty)$. □

Exercise 6. *Let $P(x, y) : x^2 + y^2 = 1$ and $Q(x, y) : x + y = 1$ be open sentences where both x and y have domain $\mathbb{Z}$. Determine the truth value of $P(x, y) \Rightarrow Q(x, y)$ for $(x, y) \in \{(1, -1), (-3, 4), (0, -1), (1, 0)\}$. Give an explanation for your answer.*

Solution.

- $(1, -1)$ We have $P(1, -1) : 1 + (-1)^2 = 1$ which is false, so the implication is true.
- $(-3, 4)$ We have $P(-3, 4) : (-3)^2 + 4^{=}1$ which is false, so the implication is true.
- $(0, -1)$ We have $P(0, -1) : 0^2 + (-1)^{=}1$ which is true, and $Q(0, -1) : 0 + (-1) = 1$ which is false, so the implication is false.
- $(1, 0)$ We have $P(1, 0) : 1^2 + 0^{=}1$ which is true, and $Q(1, 0) : 1 + 0 = 1$ which is true, so the implication is true.

□

Exercise 7. *Let D denote the set of odd integers and let*

$$P(x) : x^2 + 1 \text{ is even.} \quad \text{and} \quad Q(x) : x^2 \text{ is even.}$$

be open sentences over the domain D. State $\forall x \in D, P(x)$ and $\exists x \in D, Q(x)$ in words.

Solution.

- $(\forall)$ For every odd integer, $x^2 + 1$ is even.
- $(\exists)$ There is an odd integer such that x^2 is even.

□

Exercise 8. *State the negations of the following quantified statements, without just putting the phrase "It is not the case that..." in front of the given phrase.*

(a) *For every rational number r, the number $\frac{1}{r}$ is rational.*

(b) *There exists a rational number r such that $r^2 = 2$.*

Solution.

(a) There is a rational number r where $\frac{1}{r}$ is irrational.

(b) For every rational number r, $r^2 \neq 2$.

□

Exercise 9. *Determine the truth value of each of the following statements. Give explanations for your answers.*

(a) $\exists x \in \mathbb{R}, x^2 - x = 0.$

(b) $\forall n \in \mathbb{N}, n + 2 \geq 2.$

(c) $\forall x \in \mathbb{R}, \sqrt{x^2} = x.$

(d) $\exists x \in \mathbb{Q}, 3x^2 - 27 = 0.$

(e) $\exists x \in \mathbb{R}, \exists y \in \mathbb{R}, x + y + 3 = 8.$

(f) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x + y + 3 = 8.$

Solution.

(a) This is true, let $x = 1$.

(b) This is true since adding a positive number increases the value.

(c) This is false, let $x = -1$, then $\sqrt{x^2} = \sqrt{1} = 1 \neq -1$.

(d) This is true, let $x = 3$.

(e) This is true, let $x = 3$ and $y = 2$.

(f) This is true. Let $x \in \mathbb{R}$, then $y = 5 - x \in \mathbb{R}$ is the correct choice of y.

□

Exercise 10. *Let A be the set of circles in the plane with center $(0,0)$ and let B be the set of circles in the plane with center $(1,1)$. Furthermore, let*

$$P(C_1, C_2) : C_1 \textit{ and } C_2 \textit{ have exactly two points in common.}$$

be an open sentence where the domain of C_1 is A and the domain of C_2 is B.

(a) *Express the following quantified statement in words:*

$$\forall C_1 \in A, \exists C_2 \in B, P(C_1, C_2). \tag{1}$$

(b) *Express the negation of the quantified statement in (1) in symbols.*

(c) *Express the negation of the quantified statement in (1) in words.*

Solution.

(a) For every circle centered at $(0,0)$, there is a circle centered at $(1,1)$ which intersects the circle at $(0,0)$ in exactly two points.

(b) $\exists C_1 \in A, \forall C_2 \in B, \neg P(C_1, C_2)$.

(c) There is a circle centered at $(0,0)$ such that no circle centered at $(1,1)$ intersects it exactly twice.

□